Integrating fallibilism and formal logic: A Peircean approach to Lakatos' methodology

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Authentic learning of argumentation and proof in mathematics requires active conjecture development rather than passive reproduction of proofs. Lakatos' approach, widely used in mathematics education research, emphasizes this active process. Although Lakatos rejects formal logic as useless for knowledge development, investigating a potential logical framework for his approach could provide insights into its logical foundations and connect fallibilist and classical views on argumentation and proof in mathematics education research. This theoretical paper explores how an epistemic-logical model, mainly based on an intuitionistic version of the Peircean first-order existential graphs, can frame Lakatos' methodology, contributing to theoretical perspectives and offering a new analytic tool for researchers.

Keywords: Lakatos' approach, epistemic-logical model, intuitionistic existential graphs, fallibilism, classical logic.

Rationale

Lakatos' approach in designing and analysing meaningful mathematical activities is widely spread in mathematics education research (e.g. Deslis et al., 2022; Stylianides et al., 2022). Even though, to the best of my knowledge, investigations on the logic that could frame this approach have never been carried out in this research area. Lakatos (1976) calls his approach "the logic of discovery," (p. 3). Still, the term *logic* is intended in that context as rational scientific enterprise inspired by informal logic, i.e. "the logic of proof or of thought-experiment or of construction" (Lakatos, 1976, p. 81), rather than as deductive logic in a formal system (Lakatos, 1976, p. 3). In this sense, trying to frame Lakatos' approach by a logical system might seem nonsense. Nevertheless, discussing this aspect could be useful for at least two reasons in mathematics education research: (1) it could help to shed light on the logical bases of this approach and situate it in the panorama of logical approaches to proof in mathematics education; (2) it could act as a bridge that allows communication between fallibilist approaches to argumentation and proof in mathematics education (e.g. Stylianides et al., 2022) and approaches that suppose mathematical knowledge to be, once established, certain and unrefutable and thus framed by classical logic (e.g. Durand-Guerrier, 2005). This theoretical paper discusses how and to what extent the epistemic-logical model presented in Asenova (2023) and deepened in Asenova (2024), extended here to first-order intuitionistic logic, can frame Lakatos' approach, also keeping the door open for considering mathematical knowledge as certain, according to classical logic. It aims to contribute to the development of theoretical and methodological perspectives. and the design of an analytic tool for research in mathematics education.

Method

Hereafter, some elements of Lakatos' approach are exposed, and the epistemic-logical model (Asenova, 2023, 2024) is presented and extended from propositional to first-order logic. Then the suitability of the model for framing Lakatos' approach from a logical viewpoint is discussed. Finally, Lakatos' lemma-incorporation method is exemplified, and the conclusions are drawn.

Lakatos' approach and the methods of discovery and verification

Lakatos (1976) challenged the traditional view that mathematical knowledge is timeless, certain, and a priori. He combined Hume's problem of induction with Popper's theory of falsification to highlight the relativity of mathematical knowledge. Lakatos (1976) provided a list of heuristically detected methods used by mathematicians in their mathematical practice for improving a conjecture and its proof through counterexamples. In this process, the type of the discovered counterexample plays a crucial role. A counterexample can be local or global; it is *local* when it affects at least one step of the proof (based on a lemma acting as an argument in the proof); it is *global* when it impacts the main conjecture and thus the conclusion itself. A counterexample can be both local and global or only one of them: if it is only local, the conclusion is not affected and only the 'guilty' argument should be changed; if it is only global, the arguments seem to work but the conclusion does not and one should search for a hidden assumption and incorporate it explicitly as a condition in the conjecture; if the counterexample is both local and global, the problematic proof step affects the conclusion and should thus be incorporated explicitly as a condition.

Lakatos' most famous example (Lakatos, 1976) is the historical refinement of the justification of Euler's formula on the relation between the number of vertices V, edges E, and faces F of a polyhedron: V-F+E=2. Lakatos presents his inquiry in a dialogic form, where a maths class discusses an informal proof of this 'naïve conjecture' provided by the teacher (it is the proof historically provided by Cauchy), introducing counterexamples (e.g. the hollow cube) that force the class to explicitly define the involved concepts and to refine the proof. The proof consists of a thought-experiment that is supposed to: remove a face of the polyhedron; stretch its surface upon a plane; section the surface following certain rules based on triangulation; remove parts of the stretched surface following rules that do not change the initial relation between V, F, and E; show that at the end the relation between V, F and E is V-F+E=2 and thus must have been so at the beginning of the procedure. Below, after introducing the epistemic-logical model, this paradigmatic example is used to exemplify the modeling of one of Lakatos' methods: lemma-incorporation.

The epistemic-logical model with quantifiers

The model, which consists of three elements —Vergnaud's (2009) concepts-in-action and theorems-in-action (Vergnaud, 2009), Duval's value of a verbalised proposition (Duval, 2007), and an adaptation of Oostra's intuitionistic propositional existential graphs (Oostra, 2022), as presented in Asenova (2024) — is extended here to predicate logic to frame the use of (counter)examples.

Concepts-in-action, theorems-in-action and the value of a verbalised proposition

Vergnaud's theory of conceptual fields (2009) aims to establish connections between the operational form of knowledge, linked to actions, and the predicative form of knowledge, linked to symbolic and linguistic expressions. In this context, two main ideas are concepts-in-action and theorems-in-action that are implicit and need to be deduced from a student's actions. Theorems-in-action are expressed as propositional sentences and are either true or false, whereas concepts-in-action are relevant or not relevant. According to Vergnaud (2009), even if someone believes a false proposition to be true, it still qualifies as a theorem-in-action. What is important is that the individual acts as if the proposition were true. This perspective focuses on the student's knowledge, regardless of how it aligns with normative standards. In Duval's framework of the meaning space of a verbalised proposition (2007), the value-dimension links the proposition to its meaning. Here we focus only on the epistemic value (e.g., obvious, absurd, possible) and the logical value (true or false) because they allow to characterise the kind of reliability that students attribute to their knowledge: the epistemic value aligns with the idea of theorem-in-action, which are based on implicit, proposition-like knowledge, while the logical value corresponds to theorems and is supported by explicit concepts.

The system of intuitionistic existential graphs with quantifiers

Starting from Peirce's diagrammatic logic of existential graphs (EGs), divided into Alpha EGs for propositional logic and Beta EGs for predicate logic, Oostra (2022) formalised a topological system of classical Alpha and Beta EGs and intuitionistic Alpha and Beta EGs. Alpha-intuitionistic EGs (Alpha-IEGs) are composed of some or all the following elements: sheet of assertion (the plane surface), propositional letters (capital letters representing the propositions), two types of closed curves: cuts, that means closed curves that 'cut' the sheet of assertion into two regions: one inside and one outside of the closed curve (Figure 1 a) or scrolls (Figure 1 c, d). Each scroll is composed of a cut with one or more loops folded inside it (e.g., single or double scroll, Figure 1 c, d). According to Peirce, a scroll is: "a curved line without contrary flexure and returning into itself after once crossing itself" (Peirce, CP 4.564). The areas on the sheet of assertion are even or odd, according to the number of curves drawn around them (Figure 1).

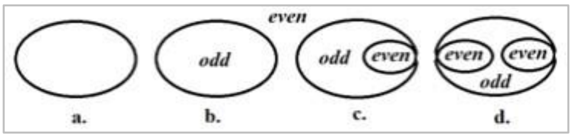

Figure 1: Elements and areas of IEGs: empty cut (a); single and double scrolls (c, d); even and odd areas (b, c, d, and the surface they are drawn upon) (Oostra, 2022, p. 136)

In a scroll, the region limited by the cut and the loops is the outer area of the scroll, and the interior part of the loop is the inner area. The scroll is a purely intuitionistic element as classical EGs work only with cuts; this is the main difference between classical and intuitionistic EGs. The interpretation of the classical and intuitionistic connectives is represented in Figure 2, where '=' stands for equivalence.

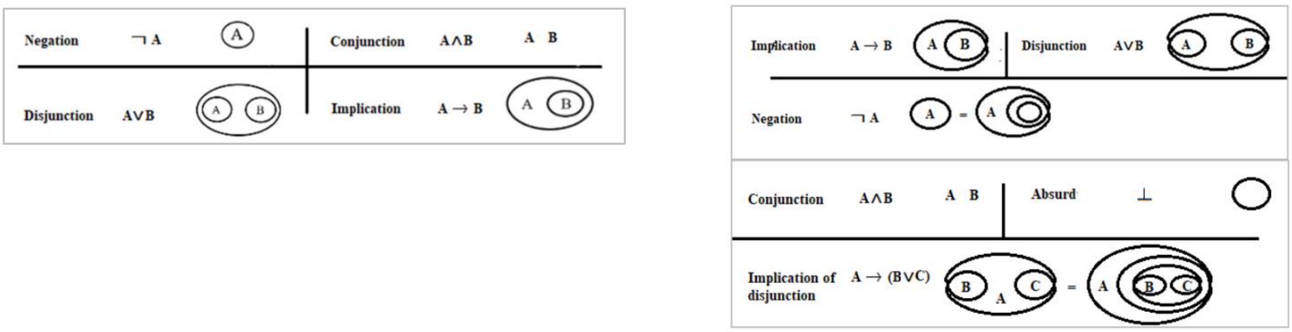

Figure 2: Basic and derived connectives for classical EGs (on the left) and intuitionistic EGs (on the right) (adapted from Oostra (2022, pp. 135–136))

According to Oostra: “we obtain the system of intuitionistic Beta graphs by adding to the Alpha system a line, which stands for a subject, element, or individual, and letters with attached lines, which stand for relatives, or predicates” (Oostra, 2022, p. 147). In this way, the system of Alpha IEGs is extended to the Beta-IEGs.

Graphically, Beta-IEGs differ from Alpha-IEGs by the use of (possibly branched) lines, called lines of identity (Figure 3a), and the use of capital letters with a positive number of (ordered) lines attached, symbolising predicates (Figure 3b); the existential quantifier corresponds to the line of identity, and the universal quantifier to the combination of the line with a scroll (Figure 3c) (Oostra, 2022).

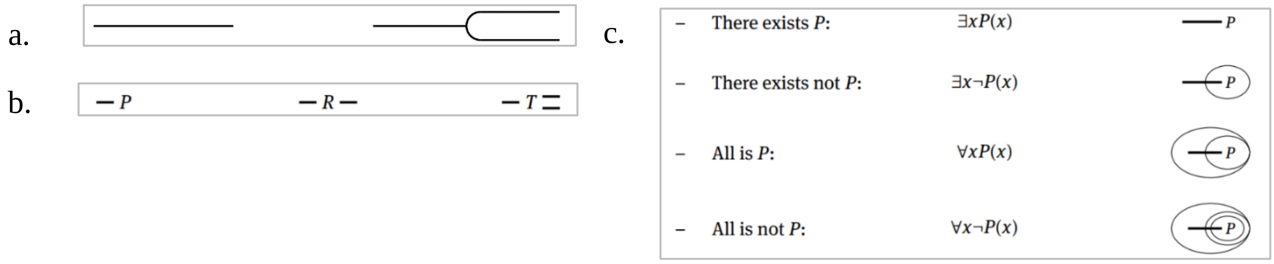

Figure 3: Existential and universal quantifier in Beta-IEGs (adapted from Oostra, 2022 (pp. 147-148))

Summing up, a Beta IEG is a diagram consisting of a finite combination of one or more of the following elements: predicate letters, cuts, scrolls, and lines of identity.

The interpretation of the intuitionistic Beta-IEGs is based on the clauses for Alpha-IEGs, with the addition of the following: drawing an identity line means asserting the existence of an individual; writing a letter with lines attached to it means that the predicate it represents holds for the involved individuals; joining two lines of identity means identifying the individuals they represent.

In Asenova (2024) the Alpha IEGs were introduced with some additional transformation rules compared to Oostra’s (2022) system, without affecting its soundness because these rules can be derived from the others. The additional rules are introduced due to the specific needs in mathematics education research which is interested in linking together aspects related to a logic concerning the epistemic value (here represented by the intuitionistic logic) and aspects related to a logical value concerning certainty (here represented by classical logic), as well as to the need to express an increasing/decreasing epistemic value. Even thought the additional transformation rules are not necessary, they are explicitly formulated as rules to render the system more user-friendly.

The lines of identity do not affect the parity of the areas. Hence, the rules of transformation that hold for Alpha IEGs hold also for Beta-IEGs, with some additions related to the line of identity. In this way, the transformation rules for Beta-IEGs can be conceived as an extension of the transformation rules for Alpha-IEGs and the latter can be conceived as an extension of the transformation rules for classical EGs. Therefore, the system of transformation rules for Beta-IEGS presented below includes the transformation rules (TRs) for classical logic (the part before the first slash in the first five rules), intuitionistic propositional logic (the part between the first end the second slash in the first five rules and rules 6 and 8), and intuitionistic predicate logic (the part after the second slash in rules from 1 to 5 and after the first slash in rules 6 and 8). TR 7 is universal and holds for all kinds of graphs. Below, ‘ITRn’ stands for ‘intuitionistic transformation rule number n.’

ITR1: *Erasure*. In an even area, any graph may be deleted/ any loop within an even area may be erased, together with its contents/ in an even area, any line of identity may be cut. ITR2. *Insertion*. In an odd area, any graph may be added/ if the odd area is limited externally by a cut, a loop containing any graph may be added to the cut/ in an odd area, two lines of identity may be joined. ITR3. *Iteration*. A graph can be repeated in its own area or in any area contained in it that is not part of the graph itself/ any loop may be iterated, together with its contents, on its own cut/ a branch with a loose end may be added to any line and any loose end of a line may be extended inwards through cuts or loops; when there are lines of identity involved in the graph to be iterated, they must correspond exactly to those of the original graph. ITR4. *Deiteration*. Any graph may be deleted if a copy of it persists in the same area or any area around it; a loop, together with its contents, may be erased if another loop with the same contents is present on its cut/ a branch with a loose end may be removed from any line; any loose end of a line may be retracted from the outside (retracted, as a reverse of iterated) through cuts or loops; when there are lines of identity involved in the graph to be deiterated, they must correspond exactly to those of the outside copy of the graph. ITR5. *Scrolling*. A scroll with an empty outer area may be drawn around or removed from any graph on any area/ The application of this rule is not prevented by the presence of lines that cross both the cut and the loop of the scroll with an empty outer area, that is, that pass from outside the scroll to the inside of the loop. ITR6. *Increasing or decreasing loop and increasing or decreasing cut*. In a scroll, the area enclosed by the loop can increase (↑) or decrease (↓) and the area of the cut can increase (↑) or decrease (↓)/ when the initial loop or scroll crosses lines, the increasing or decreasing loop or scroll must keep crossed the same lines. ITR7. *Topological equivalence*. A graph can be transformed into another equivalent by continuously transforming the curves it is composed of. ITR8. *Detachment*. A loop can be detached from the cut/ the detached loop must keep crossed the same lines as before.

The ITRs from 1 to 5 are the original rules formulated by Oostra (2022), while the ITRs 6, 7 and 8 are the rules added for the purposes in mathematics education research. With respect to Asenova (2024), ITR 6 is extended by introducing increasing/ decreasing cuts; this can be justified by ITR7.

Framing Lakatos' approach by the epistemic-logical model with quantifiers

One of the main aspects that makes the model described above useful for framing Lakatos' approach is that it is based on intuitionistic existential graphs which see classical existential graphs as a special case. Indeed, the main difference between intuitionistic and classical EGs is the meaning of the universal quantifier within the corresponding first-order logic.

The graph in Figure 4a represents the intuitionistic universal quantifier referred to the predicate P and its meaning is 'All is P;' in Figure 4b the classical universal quantifier $\forall x P(x)$ is represented; in Figure 4c the negation of 'All is P,' i.e., 'All is not P,' is represented.

Figure 4: Graphs representing the universal intuitionistic quantifier (a), the universal classical quantifier (b), the negation of the universal intuitionistic quantifier (c), and the steps to switch from 'All is not P' to 'There are P' (d)

In intuitionistic logic 'All is not P' does not imply that there are not objects satisfying P, as this is the case in classical logic, where $\forall x \neg P(x)$ implies $\neg \exists x P(x)$ ('$\neg$' represents classical negation and '$\exists$' represents the classical existential quantifier). Indeed, in intuitionistic logic the statement 'There are P' can be derived from the statement 'All is not P' (Figure 4d). This means that the judgment on the domain of a conjecture is formulated 'until proven otherwise.' In this sense, even if at a given moment it seems that a conjecture fails because it is falsified, one could still detect a safety domain where it holds. This aspect is captured by intuitionistic logic and is graphically represented by the existential graphs. Vice versa, in classical logic, the statement '$\exists x P(x)$' cannot be derived from $\forall x \neg P(x)$, as they are contradictory. Summing up, intuitionistic universal quantification keeps open the possibility of increasing or decreasing the domain of a conjecture, as it refers only to the already examined domain of the underlying concept, which could be altered if counterexamples or previously unconsidered examples are discovered.

Furthermore, the model can frame the epistemic value assigned by a subject (obvious, absurd, possible, probable, etc.) to a statement, suspending the judgment on its logical value and expressing his/her increasing or decreasing convincement (ITR6). Nevertheless, the model does not require to agree with the idea that mathematical theorems are never definitely true or false, embracing a fallibilist position on mathematical knowledge. Indeed, it keeps the door open to refutation, as described above, but also allows to shift to certainty. According to the system of IEGs presented in the previous section, switching from the graph represented in Figure 4a to the one in Figure 4b is allowed (ITR8), but not vice versa. This does not contradict Lakatos' position: even if it is generally fallibilist, it does not deny the existence of statements that are undoubtedly true (e.g., *The sum of the angles of a triangle in plane geometry is 180°*), but highlights that they belong to "mature theories," while the fallibilist approach refers to theories that are "under construction" (Lakatos, 1976, p. 42).

Mathematics taught at school (and to a large extent at university) belongs to mature theories (e.g. Euclidean Geometry or Calculus), at least from the expert's point of view. Nevertheless, from the student's point of view, this does not need to be the case: when students formulate a conjecture and try to prove it or try to identify a safety domain for it, they might be more or less convinced that it is right (according to Duval's distinction, they assign an epistemic value to the statement rather than a logical one). If (counter)examples or exceptions arise, they must be considered revising the definition of the involved concepts, the conjecture, or the just provided proof. Usually, the teacher knows the largest domain of the conjecture and the appropriate definition of the involved concept that makes the proof work. Once this domain is achieved, she can support students in shifting from the epistemic to the logical value. Summing up, from a logical point of view, the presented epistemic-logical model reconciles a perspective based on a constructive approach, interested in conjecture formulation and discovery, and based on epistemic aspects, with an approach based on truth-valued logic.

The lemma-incorporation method represents, according to Lakatos, the core of his proof-and-refutation method. It allows for the improvement of the proof itself by incorporating a local counterexample that refutes a proof step, i.e. a lemma. According to the polyhedra example proposed by Lakatos, the result of the lemma-incorporation method is that a polyhedron must be simple and with simple connected faces. This allows for the incorporation of two 'guilty' lemmas that undermine the proof. Lakatos states that this method "virtually *summed up the proof in the lemma-incorporating* *theorem*" (Lakatos, 1976, p. 42, emphasis in original). The sequence of Beta-IEGs in Figure 7 shows how the basic idea of lemma-incorporation can be modelled.

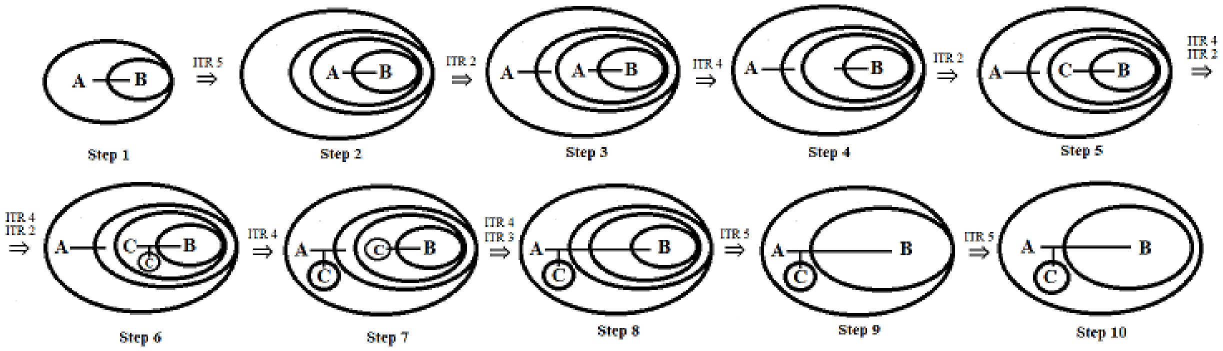

Figure 7: Modelling of the lemma-incorporation method by the epistemic-logical model

In Figure 7, Step 1 represents the statement 'All A are B,' i.e., 'All simple polyhedra are Eulerian.' Steps from 2 to 5 are technical steps needed to introduce a lemma (C) used in the proof. In this way, in step 5, we have the statement 'All A are B because of C', that means 'All simple polyhedra (A) are Eulerian (B) because, in the triangulating process, by drawing a new diagonal edge, one always increases by one the number of edges and faces (C)'. According to Lakatos (1976), this lemma becomes suspicious as it turns out that in the case of the crested cube (a cube with a smaller cube setting on top of it), one needs to increase the edges by two to increase the number of faces by one. Step 6 represents the 'appearance' of the counterexample, i.e., the crested cube, which is a simple polyhedron but for which the lemma does not work. The counterexample is represented by the negation of C: a cut with the propositional letter 'C' inside it. In step 7, the lemma is turned into the condition 'polyhedra for which any face dissected by a diagonal edge falls into two pieces,' equivalent to the 'guilty' lemma. Steps 8 and 9 are technical steps that show how the graphs evolve to the final statement in step 9 ('All simple polyhedra for which any face dissected by a diagonal edge falls into two pieces are Eulerian'), while step 10 represents the possible shift to certainty and classical logic.

The lemma-incorporation method leads to consider the interplay of concepts-in-action (e.g., the concept of simple polyhedron) and theorems-in-action (the incorporated lemma) and their evolution to explicit concepts and theorems. Indeed, step 10 represents the possible shift from a theorem-in-action to a theorem and thus from an epistemic to a logical value, showing how the first two elements of the theoretical framework are used to interpret the third one.

Conclusions

In this paper, the suitability of the presented epistemic-logical model in framing Lakatos' approach to proof and refutations (Lakatos, 1976) was discussed and exemplified. From a theoretical viewpoint, this allows us to situate Lakatos' approach in the panorama of logical perspectives on argumentation and proof in mathematics education, associating it to a constructivist logic that focuses on processes rather than on states of affairs. From a didactic point of view, this analytical tool can be used by the researchers to better frame student's reasoning when it represents aspects difficult to frame with classical logic. Indeed, the technical part of the model allows to symbolically represent the reasoning similarly as this is done by symbolic logic, but the topological form of the representation allows us accounting for continuities and discontinuities in the reasoning process and for growing and decreasing epistemic values, as it is shown in the examples in Asenova (2023, 2024). Furthermore, it accounts for shifts between an epistemic and a logical value, and thus for shifts from implicit theorems-in-actions to theorems, as shown in the example in figure 7. As the model involves not only a fallibilist but also a truth-value perspective, it helps to bridge communication between fallibilist approaches to argumentation and proof (e.g. Stylianides et al., 2022) and approaches that suppose mathematical knowledge to be, once established, certain and unrefutable (e.g., Durand-Guerrier, 2005). Further research should be done to deepen the model's potential in bridging this gap and in using it for the design of tasks for teacher training courses.

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